PUTNAM PRACTICE SET 1

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Problem 1. Find the maximum and the minimum possible value of the product $x_1 \cdot x_2 \cdots x_n$, where the real numbers x_i satisfy the following properties:

- $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$; and
- $x_i \ge \frac{1}{n}$ for each $i = 1, \dots, n$.

Solution. We use the inequality between the Arithmetic Mean and the Geometric Mean and therefore, conclude that

$$\sqrt[n]{x_1^2 \cdot x_2^2 \cdots x_n^2} \le \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} = \frac{1}{n}$$

with equality if and only if $x_1^2 = x_2^2 = \cdots = x_n^2 = \frac{1}{n}$. So, the maximum of $\prod_{i=1}^n x_i$ is $\frac{1}{n^{\frac{n}{2}}}$ and it is attained when $x_1 = x_2 = \cdots = x_n = \frac{1}{\sqrt{n}}$ (which is allowed since $\frac{1}{\sqrt{n}} > \frac{1}{n}$).

Now, in order to determine the minimum of $\prod_{i=1}^n x_i$ we use the following easy claim.

Claim 0.1. *Let u, v, u_1, v_1 be positive real numbers such that*

- $u + v = u_1 + v_1$; *and*
- $\min\{u, v\} \ge u_1$.

Then $uv \ge u_1 v_1$.

Proof of Claim 0.1. Using the above hypothesis, we have that $v_1 \ge u_1$; also, without loss of generality, we may assume $v \ge u$. Then (because $u_1 \le u \le v \le v_1$) we have

$$v_1 - u_1 \ge v - u$$

and so, $4u_1v_1 = (u_1 + v_1)^2 - (v_1 - u_1)^2 \le (u + v)^2 - (v - u)^2 = 4uv$, thus proving the desired claim. □

So, when we minimize $\prod_{i=1}^n x_i^2$ with $x_n = \max_{i=1}^n x_i$, then for any $i = 1, \dots, n-1$, we may replace x_1 by $x_1' := \frac{1}{n}$ and then replace x_n by $x_n' := \sqrt{x_1^2 + x_n^2 - \frac{1}{n^2}}$—this will only decrease the above product. So, the minimum of $\prod_{i=1}^n x_i$ is obtained for

$$x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{n} \text{ and } x_n = \sqrt{\frac{n^2 - n + 1}{n^2}}$$

and so, $\min \prod_{i=1}^n x_i = \frac{\sqrt{n^2-n+1}}{n^n}$.

Problem 2. We let $f : [0, 1) \longrightarrow [0, 1)$ be defined by the properties:

$$\begin{cases} f(x) = \frac{f(2x)}{4} & \text{if } 0 \le x < \frac{1}{2} \\ f(x) = \frac{3+f(2x-1)}{4} & \text{if } \frac{1}{2} \le x < 1 \end{cases}.$$

Find $f(x)$ for each $x \in [0,1)$; you may express your answer in terms of the expansion of x in base 2.

Solution. We write $x = 0.b_1b_2\cdots b_n\cdots$ in base 2, i.e., $b_n \in \{0,1\}$ for all $n \geq 1$. Also, we use the convention that we write

$$0.b_1\cdots b_m100\cdots0\cdots$$

and not $0.b_1\cdots b_m011\cdots1\cdots$. Then our definition for $f(x)$ yields the following:

- if $b_1 = 0$ then $f(x) = \frac{f(0.b_2b_3\cdots b_n\cdots)}{4}$, while
- if $b_1 = 1$ then $f(x) = \frac{3+f(0.b_2b_3\cdots b_n\cdots)}{4}$.

In both cases, we get

$$f(x) = \frac{b_1}{2} + \frac{b_1}{4} + \frac{f(0.b_2b_3\cdots b_n\cdots)}{4}$$

and then inductively, we obtain

$$f(x) = \frac{b_1}{2} + \frac{b_1}{4} + \frac{b_2}{8} + \frac{b_2}{16} + \frac{f(0.b_3b_4\cdots b_n\cdots)}{16}$$

and furthermore, for any $m \geq 1$, we have

$$f(x) = 0.b_1b_1b_2b_2\cdots b_{m-1}b_{m-1}b_mb_m + \frac{f(0.b_{m+1}b_{m+2}\cdots)}{4^m}.$$

Now, since $f(z) < 1$ for each z, then

$$\frac{f(0.b_{m+1}b_{m+2}\cdots)}{4^m} \to 0 \text{ as } m \to \infty$$

and so, we conclude that

$$f(0.b_1b_2\cdots b_n\cdots) = 0.b_1b_1b_2b_2\cdots b_nb_n\cdots$$

Problem 3. Find all real numbers a for which there exist nonnegative real numbers $x_1,\dots,x_5$ satisfying the following property:

$$\sum_{k=1}^{5} k^{2i-1}\cdot x_k = a^i \text{ for each } i = 1,2,3.$$

Solution. The given relations yield

$$\sum_{k=1}^{5}(ak-k^3)\cdot x_k = 0 \text{ and } \sum_{k=1}^{5}(ak^3-k^5)x_k = 0.$$

So,

$$\sum_{\substack{1\leq k\leq 5\\ k^2\leq a}} (a-k^2)kx_k = \sum_{\substack{1\leq k\leq 5\\ a<k^2}} (k^2-a)kx_k$$

and

$$\sum_{\substack{1\leq k\leq 5\\ k^2\leq a}} (a-k^2)k^3x_k = \sum_{\substack{1\leq k\leq 5\\ a<k^2}} (k^2-a)k^3x_k.$$

However,

$$\begin{aligned}
&\sum_{\substack{1\le k\le 5\\ k^2\le a}} (a-k^2)k^3x_k\\
\le &\sum_{\substack{1\le k\le 5\\ k^2\le a}} (a-k^2)\cdot a\cdot kx_k\\
= &\sum_{\substack{1\le k\le 5\\ a<k^2}} (k^2-a)kx_k\cdot a\\
\le &\sum_{\substack{1\le k\le 5\\ a<k^2}} (k^2-a)k^3x_k
\end{aligned}$$

and since the first and the last of these sums are equal, then it must be that both inequalities above are actually equalities. Now, if $a\notin\{1^2,2^2,3^2,4^2,5^2\}$ then the above inequalities cannot become equalities (since not all of the x_k can be equal to zero). On the other hand, for each $m\in\{1,\dots,5\}$, if $a=m^2$ then letting $x_k=0$ if $k\ne m$ while $x_m=m$, then all of the hypotheses are met.

Problem 4. Let $m\in\mathbb{N}$ and let $a_1,\dots,a_m\in\mathbb{N}$. Prove that there exists a positive integer $n<2^m$ and there exist positive integers $b_1,\dots,b_n$ satisfying the following properties:

(i) for any two distinct subsets $I,J\subseteq\{1,\dots,n\}$, we have that $\sum_{k\in I}b_k\ne\sum_{\ell\in J}b_\ell$; and
(ii) for each $i=1,\dots,m$, there exists a subset $J_i\subseteq\{1,\dots,n\}$ such that $a_i=\sum_{k\in J_i}b_k$.

Solution. We write for each $i=1,\dots,m$:

$$a_i=\sum_{j\in M_i}2^j,$$

where M_i is the set of nonnegative integers corresponding to the positions in the writing of a_i in base 2 where the digit of a_i equals 1. Writing similarly each b_i (for $1\le i\le n$)

$$b_i=\sum_{j\in A_i}2^j,$$

then the above conditions (i)-(ii) are satisfied if the following conditions are met:

(i') $A_i\cap A_j=\emptyset$ if $1\le i<j\le n$; and
(ii') for each $i=1,\dots,m$, there exists $J\subset\{1,\dots,n\}$ such that $M_i=\cup_{j\in J}A_j$.

Furthermore, we need $n\le 2^m-1$.

Now, we prove the existence of such sets $\{A_j\}_{1\le j\le n}$ corresponding to given sets $\{M_i\}_{1\le i\le m}$ (with $n\le 2^m-1$). We argue by induction on m; the case $m=1$ is immediate since we may take $n=1<2^1$ and $A_1:=M_1$.

Next, we assume we constructed $A_1,\dots,A_n$ (with $n\le 2^m-1$) corresponding to the sets $M_1,\dots,M_m$ and given a new set M_{m+1}, we construct the following sets (note that if one of these sets A_i' is empty, then we could simply disregard the corresponding a_i'):

$$A_{2n+1}' := M_{m+1} \setminus \left(\bigcup_{i=1}^{n} A_i \right)$$

$$A_{2i}' := M_{m+1} \cap A_i \text{ for } 1 \le i \le n \text{ and}$$

$$A_{2i-1}' := A_i \setminus M_{m+1} \text{ for } 1 \le i \le n.$$

Since $A_i = A_{2i}' \cup A_{2i-1}'$ for $1 \le i \le n$, then each M_j (for $1 \le j \le m$) can be written as a union of the sets A_i' (for $1 \le i \le 2n$). Also, since $A_i \cap A_j = \emptyset$ if $1 \le i < j \le n$, then $A_i' \cap A_j' = \emptyset$ if $1 \le i < j \le 2n$. Furthermore,

$$M_{m+1} = \bigcup_{i=1}^{2n+1} A_i'$$

and also, A_{2n+1}' is disjoint from each A_i' for $1 \le i \le 2n$. So, the hypotheses are met for the sets A_i' for $1 \le i \le 2n+1$ and clearly,

$$2n + 1 \le 2(2^m - 1) + 1 = 2^{m+1} - 1.$$